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POISSON BOUNDARIES OF DISCRETE GROUPS OF MATRICES

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ABSTRACT

If μ is a probability measure on a countable group there is defined a notion of the Poisson boundary for μ which enables one to represent all bounded μ-harmonic functions on the group. It is shown that for discrete groups of matrices this boundary can be identified with the boundary of the corresponding Lie group.

The behavior at infinity of a countable group G is partly described by boundaries. We consider here Poisson boundaries of random walks on T: Let μ be a probability measure on G, and call a function h on G μ-harmonic when for any g in G,

$$h(g)=\sum_{g'\in G} h(gg')\mu(g').$$

A Poisson boundary is a compact probability space on which G acts and which represents all bounded harmonic functions by a formula analogous to the Poisson representation of harmonic functions on the disk. (See [5] and [8] for a detailed recent study.)

The Poisson boundary reflects properties of the group itself: for instance, it is trivial if G is abelian [2] or nilpotent [4], but examples show that it can be non-trivial even when G is amenable [8]. It can also be described when G is the free group with k generators [4]. It also reflects how the group can be imbedded in other groups: Furstenberg proved that if G is cocompact in $\mathrm{SL}(d,\mathbf{R})$, there exists a probability measure on G such that the Poisson boundary is the Furstenberg boundary with its natural probability measure (see [5]). These results extend to lattices in semi-simple Lie groups and to other probabilistic questions (see [6], [11], [7]).

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Using a different approach, Series [14] could describe the Poisson boundary (and even the Martin boundary, corresponding to positive harmonic functions) for a large class of Fuchsian groups.

Here we consider a general discrete group of invertible real square ($d \times d$) matrices and following one of Furstenberg's approaches (see [7], [3], [8]) we use an entropy criterion. We shall require a boundedness condition, namely

$$\sum_{g \in G} \log \| g \| \mu(g) < +\infty, \qquad \sum_{g \in G} \log \| g^{-1} \| \mu(g) < +\infty,$$

and a non-degeneracy condition, namely that the semi-group generated by supp μ is the whole G. There is a natural boundary in this case, which is some quotient of the space of d-dimensional flags, and a natural invariant measure on it, both defined by using Lyapunov exponents of the random walk. We shall prove here that this boundary is a Poisson boundary, thus recovering some geometry of G only by looking at it as an abstract group. Like in [9], where we proved the same result for SL(2, C), we use heavily Oseledeč theorem to compare entropies with dimensional quantities on the boundary.

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I. Notations and results

I.1. The natural boundary

Let μ be a probability measure on a locally compact separable group G. Let S be a compact metric space and $(g, x) \to g \cdot x$ a continuous action of G on S ($e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$). If ρ is a measure on S, let $g \cdot \rho$ denote the measure defined by $g \cdot \rho(f) = \int f(g \cdot x) \rho(dx)$ for all continuous real functions f on S. We shall call (S, ρ) a (G, μ)-space if ρ is an invariant probability measure, i.e., if $\int g \cdot \rho(f) \mu(dg) = \rho(f)$ for all continuous f.

Let us consider the product space $(\Omega, \mathbf{P})$ of an infinite sequence of copies of (G, μ). The (G, μ) space (S, ρ) is called a *boundary* if for $\mathbf{P}$-a.e. $\omega = \{g_1, g_2, \ldots, g_n, \ldots\}$ the sequence of measures ρ_n converges weakly towards some Dirac measure $\delta_{Z(\omega)}$, where $\rho_n = g_1 \cdot g_2 \cdot \cdots \cdot g_n \cdot \rho$. If it is the case, it is clear by Lebesgue dominated convergence theorem that the law of the variable Z on S is ρ, which means the following relation: $\int f(Z(\omega)) \mathbf{P}(d\omega) = \rho(f)$ for all continuous functions f on S.

We suppose from now on that G is a subgroup of $\mathrm{GL}(d,\mathbf{R})$ and

$$\int \log \| g \| \mu(dg) < +\infty, \qquad \int \log \| g^{-1} \| \mu(dg) < +\infty.$$

Denote $\Lambda^k g$ the operator acting on $\Lambda^k \mathbf{R}^d$ canonically associated to g. The *exponents* are reals numbers $\lambda_1 \geqq \cdots \geqq \lambda_d$ such that

$$\sum_{i=1}^{k} \lambda_i = \lim_{n\to\infty} \frac{1}{n} \int \log \left\| \overset{k}{\Lambda} g \right\| \mu^{(n)}(dg),$$

where $\mu^{(n)}$ denotes the n-fold convolution of μ, $\mu^{(n)} = \mu^{(n-1)} * \mu$, $n = 2, \ldots$ (see, for instance, [10]). The norm $\| \quad \|$ will always be the euclidean operator norm.

Let us denote P the group of matrices in $\mathrm{GL}(d,\mathbf{R})$ with $p_{ij} = 0$ when $\lambda_i < \lambda_j$ and $B = \mathrm{GL}(d,\mathbf{R})/P$ the corresponding homogeneous space. The space B is compact metric and G acts continuously by left multiplication.

Let $0 < j_1 < j_2 < \cdots < j_r = d$ be the indices with $\lambda_{j_i} > \lambda_{j_i+1}$, $i = 1, \ldots, r-1$. Points in B are in one-to-one correspondence with the following filtrations of $\mathbf{R}^d$, $\{0\} \subset E_1 \subset E_2 \subset \cdots \subset E_r = \mathbf{R}^d$ with $\dim E_i = j_i$, $i = 1, \ldots, r$ and G acts naturally on this representation. A matrix v belongs to the class of b iff its column vectors v_j satisfy: $v_1, \ldots, v_{j_i}$ generate E_i, $i = 1, \ldots, r$.

The following "cocycle" does not depend on the choice of the matrix v in the class of b:

$$\sigma_i(g,b) = \log \frac{\left\| \bigwedge_{k=1}^{j_i} g v_k \right\|}{\left\| \bigwedge_{k=1}^{j_i} v_k \right\|}, \qquad i = 1, \ldots, r.$$

PROPOSITION 1. *There exists a unique invariant measure ν on B such that*

$$\int\int \sigma_i(g,b)\nu(db)\mu(dg) = \sum_{k=1}^{j_i} \lambda_k, \qquad i = 1, \ldots, r.$$

The (G,μ) space (B,ν) is a (G,μ)-boundary.

We call the (G,μ) space (B,ν) of Proposition 1 the *natural boundary*.

A measurable bounded function h on G is called μ-*harmonic* if $h(g) = \int h(gg')\mu(dg')$ for all g in G. We consider the space $\mathcal{H}$ of μ-harmonic functions, with uniform norm. A (G,μ) space (S,ρ) is called a *Poisson boundary* if there is an isometry $u : \mathcal{H} \to L^\infty(S,\rho)$ such that all harmonic functions have the following Poisson representation:

$$h(g) = \int_S u(h)(x) \frac{dg\rho}{d\rho}(x)\rho(dx).$$

Our main result is the following:

THEOREM A. *Let G be a discrete subgroup of* $\mathrm{GL}(d,\mathbf{R})$, *$\mu$ a probability measure on G satisfying* $\Sigma_G \log \| g \| \mu(g) < +\infty$, $\Sigma_G \log \| g^{-1} \| \mu(g) < +\infty$, *and* $\bigcup_n \operatorname{supp} \mu^{(n)} = G$.

Then the natural boundary (B,ν) is a Poisson boundary.

In particular we can specify:

THEOREM B. *Let (G,μ) be as in Theorem A. Suppose all exponents coincide $\lambda_1 = \lambda_d$. Then all bounded harmonic functions are constant.*

I.2. Entropy

The proof of Theorem A uses several notions of entropy.

First let us consider a countable group G and a probability measure μ. We put $H(\mu) = -\Sigma_G \mu(g) \log \mu(g)$ and if $H(\mu) < +\infty$,

$$h(G,\mu) = \inf_n \frac{1}{n} H(\mu^{(n)}) \qquad \text{(see [1])}.$$

Suppose G is a discrete group of matrices, μ a measure such that $\Sigma_G \log \| g \| \mu(g) < +\infty$, $\Sigma_G \log \| g^{-1} \| \mu(g) < +\infty$. We shall estimate $h(G,\mu)$ through some geometric quantity on the natural boundary (B,ν). We need some technical definitions. Consider the sequence $\lambda_1 \geqq \cdots \geqq \lambda_d$ exponents and let n, δ be positive real numbers. Two orthogonal matrices k and k' are said to be (n,δ) close if the general entry u_{ij} of $k^{-1}k'$ is such that for all distinct values λ, μ

$$\sum_{\{(i,j) \mid \lambda_i = \lambda, \lambda_j = \mu\}} | u_{i,j} |^2 \leqq \exp(-2n(|\lambda - \mu| - \delta)).$$

Two points b and b' are said to be (n,δ) close if all orthogonal matrices k and k' in the class of b and b' respectively are (n,δ) close. The property depends clearly on b and b' in a symmetric way. We denote $U_{n,\delta}(b)$ the set of points in $B(n,\delta)$ close to b, we call

$$\phi(B,\nu,\delta,\varepsilon) = \inf\left\{ t \,\middle|\, \liminf_{q\to\infty} \nu(E(t,q,\delta)) \geqq 1-\varepsilon \right\}$$

where

$$E(t,q,\delta) = \{ b \mid \nu(U_{q,\delta}(b)) \geqq \exp - qt \}$$

and $\phi(B,\nu) = \lim_{\delta \searrow 0} \lim_{\varepsilon \searrow 0} \phi(B,\nu,\delta,\varepsilon)$.

In other words $\phi(B,\nu)$ is the best upper estimate for the limit in measure of $-(1/q)\log \nu(U_{q,0}(b))$. (If $d=2$, $\lambda_1 \neq \lambda_2$, we know that the sequence $-(1/q)\log \nu(U_{q,0}(b))$ converges in measure [10].)

PROPOSITION 2. *Let G be a discrete group of matrices, μ a probability measure on G, with*

$$\sum_G \log \| g \| \mu(g) < +\infty, \qquad \sum_G \log \| g^{-1} \| \mu(g) < +\infty,$$

(B,ν) be the natural boundary, then

$$h(G,\mu) \leqq \phi(B,\nu).$$

On the other hand, let us consider a probability μ on a locally compact group G and (S,ρ) a (G,μ)-space. We call entropy of (S,ρ) the quantity $\alpha(S,\rho)$,

$$\alpha(S,\rho) = -\int_{S\times G} \log \frac{dg^{-1}}{d\rho}\, \rho(x)\rho(dx)\mu(dg).$$

The entropy $\alpha(S,\rho)$ is positive, finite or infinite.

PROPOSITION 3. *Let μ be a probability on* GL$(d,\mathbf{R})$, *with $\int \log \| g \| \mu(dg) < +\infty$, $\int \log \| g^{-1} \| \mu(dg) < +\infty$ and (B,ν) be the natural boundary. Then*

$$\phi(B,\nu) \leqq \alpha(B,\nu).$$

Propositions 2 and 3 are the two entropy estimates we shall prove. Theorem A is then a clear consequence of Propositions 2, 3 and 4:

PROPOSITION 4. *Let G be a countable group and μ a probability measure on G with $H(\mu) < +\infty$ and $\bigcup_n \operatorname{supp} \mu^{(n)} = G$. A (G,μ)-space (S,ρ) is a Poisson boundary as soon as*

(i) *(S,ρ) is a boundary,*

(ii) *$\alpha(S,\rho) \geqq h(G,\mu)$.*

Proposition 4 is due to Vershik and Kaimanovich [8] (see section 3.2; see also [3]).

We shall first recall Oseledeč' theorem and prove Proposition 1 by constructing the natural boundary out of the filtration it gives. The proof of Propositions 2 and 3 will then consist in translating geometrically on $\phi(B,\nu)$ the estimates given by Oseledeč' theorem.

Remark now that if $\lambda_1 = \lambda_d$, $P = G$ and B is reduced to a point. Any two orthogonal matrices are (n,δ) close and $\phi = 0$. Proposition 3 is then trivial but

Proposition 2 makes sense and says that $h(G,\mu)=0$ in this case. Thus for a direct proof of Theorem B, section 1.3 and 2 are not needed and the arguments in section 3 are simpler.

I.3. Oseledeč' theorem and proof of Proposition 1

Let μ be a probability measure on $GL(d,\mathbf{R})$, with $\int \log \| g^{-1} \| \mu(dg) < +\infty$, $\int \log \| g \| \mu(dg) < +\infty$ and denote $(\bar{\Omega}, \bar{\mathbf{P}})$ the infinite product of $\mathbf{Z}$-sequences of copies of $(GL(d,\mathbf{R}),\mu)$, θ the inverse shift transformation: $(\theta\bar{\omega})_i = \bar{\omega}_{i-1}$, $i \in \mathbf{Z}$, and A the matrix $A(\bar{\omega}) = (\bar{\omega}_{-1})^{-1}$.

The exponents of the system $(\bar{\Omega}, \bar{\mathbf{P}}, \theta, A)$ are the numbers $-\lambda_d \geqq -\lambda_{d-1} \geqq \cdots \geqq -\lambda_1$. By Oseledeč' theorem, there exists a subset E_1 of $\bar{\Omega}$, $\mathbf{P}(E_1)=1$, such that if $\bar{\omega} \in E_1$ and if we write a Cartan decomposition of the matrix $\bar{\omega}_{-n}^{-1} \cdots \bar{\omega}_{-1}^{-1} = L_n \Delta_n K_n$, where L_n and K_n are orthogonal, and Δ_n diagonal with increasing diagonal entries $\delta_1^{(n)}(\bar{\omega}) \leqq \cdots \leqq \delta_d^{(n)}(\bar{\omega})$, then:

(i) $$\lim_{n\to+\infty} \frac{1}{n} \log \delta_j^{(n)}(\bar{\omega}) = -\lambda_j, \qquad j=1,\ldots,d.$$

(ii) Let $K(\bar{\omega})$ be any limit point of the sequence $K_n(\omega)$, then for all δ, $K_n^{-1}(\bar{\omega})$ is (n,δ) close to $K^{-1}(\bar{\omega})$ for n large enough.

(iii) Let $b(\bar{\omega})$ be the class in B of $K^{-1}(\bar{\omega})$. This is the only point b in B such that for any matrix v in the class of b, and for $i=1,\ldots,r$,

$$\lim_{n\to+\infty} \frac{1}{n} \left\| \log\left(\overset{j_i}{\wedge} (\bar{\omega}_{-n}^{-1} \cdots \bar{\omega}_{-1}^{-1})\right)(v_1 \wedge \cdots \wedge v_{j_i}) \right\| = -\sum_{j=1}^{j_i} \lambda_j.$$

Some of these limits are strictly bigger for $b \neq b(\bar{\omega})$.

(iv) The map $b : \bar{\Omega} \to B$ is measurable with respect to the σ-algebra generated by the coordinate maps $\bar{\omega} \to \bar{\omega}_{-i}$, $i>0$.

(v) There exists a measurable decomposition of $\mathbf{R}^d$, $\mathbf{R}^d = W^1(\bar{\omega}) \oplus W^2(\bar{\omega}) \oplus \cdots \oplus W^r(\bar{\omega})$ such that a vector $v \neq 0$ belongs to $W^i(\bar{\omega})$ iff

$$\lim_{n\to+\infty} \frac{1}{n} \log \| \bar{\omega}_n \cdots \bar{\omega}_0 v \| = \lambda_{j_i}$$

and

$$\lim_{n\to+\infty} \frac{1}{n} \log \| \omega_{-n}^{-1} \cdots \bar{\omega}_{-1}^{-1} v \| = -\lambda_{j_i}.$$

Up to notations, Oseledeč' theorem is proved in this precise form in [12] (see also [13] and [10] §I.3). Remark that here the diagonal entries in Δ_n are increasing and that is why the column vectors of $K^{-1}(\bar{\omega})$ give the Oseledeč filtration when read from left to right (and not from right to left as usually when the entries of Δ_n are decreasing).

Let us choose once and for all in a measurable way a matrix $w(\bar{\omega})$ with column vectors $w_k(\bar{\omega})$, $k=1,\dots,d$ such that the vectors $w_k(\bar{\omega})$, $k=j_{i-1}+1,\dots,j_i$ form an orthonormal basis of $W^i(\bar{\omega})$. A matrix v belongs to the class $b(\bar{\omega})$ iff there exists some p in P such that $v=w(\bar{\omega})p$.

We now use this result to prove Proposition 1. When proving Propositions 2 and 3, we shall keep in mind the construction and the properties of the natural boundary that we describe now.

We consider $\hat{\theta}:\bar{\Omega}\times B$ defined by

$$\hat{\theta}(\bar{\omega},b)=(\theta^{-1}\bar{\omega},\bar{\omega}_0 b)$$

and the functions $\bar{\sigma}_i(\bar{\omega},b)=\sigma_i(\bar{\omega}_0,b)$.

LEMMA 1. *The measure* $\bar{M}(d\bar{\omega},db)=\bar{\mathbf{P}}(d\bar{\omega})\delta_{b(\bar{\omega})}(db)$ *is the unique* $\hat{\theta}$-*invariant measure on* $\bar{\Omega}\times B$ *which projects onto* $\bar{\Omega}$ *into the probability* $\bar{\mathbf{P}}$ *and satisfies*:

$$\int\bar{\sigma}_i(\bar{\omega},b)\bar{M}(d\bar{\omega},db)=\sum_{j=1}^{j_i}\lambda_j,\qquad i=1,\dots,r.$$

PROOF OF LEMMA 1. If $\bar{M}$ is any $\hat{\theta}$-invariant probability measure on $\bar{\Omega}\times B$, which projects into $\bar{\mathbf{P}}$, we have for $\bar{M}$ a.e. $(\bar{\omega},b)$ any matrix v in the class of b, any $i=1,\dots,r$:

$$\begin{aligned}\lim_n\frac{1}{n}\log\left\|\left(\bigwedge^{j_i}(\bar{\omega}_{-n}^{-1}\cdots\bar{\omega}_{-n}^{-1}\right)(v_1\wedge\cdots\wedge v_{j_i})\right\|&=\lim_n\frac{1}{n}\sigma_i(\bar{\omega}_{-n}^{-1}\cdots\bar{\omega}_{-1}^{-1},b)\\&=\lim_n-\frac{1}{n}\sum_{k=1}^{n}\sigma_i(\hat{\theta}^{-k}(\bar{\omega},b))\\&=\alpha_i(\bar{\omega},b)\end{aligned}$$

with $\int\alpha_i(\bar{\omega},b)\bar{M}(d\bar{\omega},db)=-\int\bar{\sigma}_i d\bar{M}$.

Property (iii) above tells that these limits can be $-\sum_{j=1}^{j_i}\lambda_j$ only if for $\bar{\mathbf{P}}$-a.e. $\bar{\omega}$, the conditional measure $\bar{M}_{\bar{\omega}}$ is carried by $b(\bar{\omega})$. The only possible measure is thus $\bar{M}(d\bar{\omega},db)=\bar{\mathbf{P}}(d\bar{\omega})\delta_{b(\bar{\omega})}(db)$.

This measure is invariant because $b(\theta\bar{\omega})=\bar{\omega}_{-1}^{-1}b(\bar{\omega})$ and consequently:

$$\begin{aligned}\int f.\hat{\theta}d\bar{M}&=\int f(\theta^{-1}\bar{\omega},\bar{\omega}_0 b(\bar{\omega}))\bar{\mathbf{P}}(d\bar{\omega})=\int f(\bar{\omega},\bar{\omega}_{-1}b(\theta\bar{\omega}))\mathbf{P}(d\bar{\omega})\\&=\int f(\bar{\omega},b(\bar{\omega}))\bar{\mathbf{P}}(d\bar{\omega})=\int fd\bar{M}.\end{aligned}$$

■

We prove now Proposition 1: consider the compact space B. By compactness there exist μ-invariant probability measures on B and for any such measure ν, we make the following construction.

Let us project $\bar{\Omega}\times B$ on $\Omega\times B$ by π, where $(\Omega,\mathbf{P})$ denotes the one-sided product space corresponding to the positive coordinates in $(\bar{\Omega},\bar{\mathbf{P}})$. There is a unique $\hat{\theta}$-invariant measure $\bar{M}$ on $(\bar{\Omega}\times B)$ such that $\bar{M}\circ\pi^{-1}=\mathbf{P}\times\nu$. Therefore by Lemma 1 there is at most one invariant measure ν such that the corresponding $\bar{M}$ satisfies $\int\bar{\sigma}_i d\bar{M}=\Sigma_{j=1}^{i}\lambda_j$.

This proves the uniqueness of ν because $\bar{\sigma}_i$ factorizes in σ_i by π. This proves also the existence of ν because by property (iv) the measure $\bar{M}\circ\pi^{-1}=(\mathbf{P}\delta_b)\circ\pi^{-1}$ is the product measure $\mathbf{P}\times\nu$ of $\mathbf{P}$ and the law ν of b.

Furthermore the actual construction of $\bar{M}$ by extending successively $\mathbf{P}\times\nu$ to all σ-algebras $\hat{\theta}^n\pi^{-1}(\mathcal{A}\otimes\mathcal{B})$ gives that the martingale of conditional measures $\bar{\omega}_{-1}\cdots\bar{\omega}_{-n}\nu$ converges $\bar{\mathbf{P}}$-a.e. towards the conditional measures of $\bar{M}$, i.e. $\delta_{b(\bar{\omega})}$. This shows that (B,ν) is a boundary, because under $\bar{\mathbf{P}}$, the sequence $\{\bar{\omega}_{-i}, i>0\}$ is also an independent sequence of matrices with common law μ.

This finishes the proof of Proposition 1. We also know that the measure ν is the law of the point $b(\bar{\omega})$ considered in Oseledeč' theorem. Remark also that by construction the systems $(\bar{\Omega}\times B,\bar{M},\hat{\theta})$ and $(\Omega\times B,\mathbf{P}\times\nu,\hat{\theta})$ are ergodic.

II. Dimension estimate

2.1. *Geometry of B*

We fix in this section a sequence $\lambda_1\geqq\cdots\geqq\lambda_d$ and consider the corresponding space B and the corresponding notion of (n,δ) closeness on B. We prove several properties of these objects; C_1, C_2, C_3 will be constants depending only on d and on the fixed sequence $\lambda_1\geqq\cdots\geqq\lambda_d$.

Let K be the group of orthogonal matrices, $M=K\cap P$; the Iwasawa decomposition of matrices $\mathrm{GL}(d,\mathbf{R})=KAN$ identifies P with MAN and B with K/M. A point b in B is thus identified with the class kM of orthogonal matrices in b.

LEMMA 2. *The points b and b' are (n,δ)-close as soon as there is some pair (k,k') of orthogonal matrices in the classes b and b' respectively, and k and k' are (n,δ)-close.*

The proof is easy: it consists in checking that if k and k' are (n,δ)-close, m and m' are matrices in M, km and $k'm'$ are also (n,δ) close. ■

A subset E of B is called (n, δ) separated if any two distinct points in E are (n, δ) apart. Recall that $U_{n,\delta}(x)$ is the set of points (n, δ) close to x.

LEMMA 3. *There exist constants C_1, C_2 such that if E is (n, δ) separated, and x any point in B, then $U_{n,\delta}(x)$ contains at most $C_1 C_2^{n\delta}$ points of E.*

PROOF. Let b and b' be points in $U_{n,\delta}(x)$ and k and k' orthogonal matrices such that

$$b = kM, \qquad b' = k'M.$$

We claim that if the general entry w_{ij} of $x^{-1}(k' - k)$ satisfies

$$|w_{ij}| \leqq \frac{1}{d} \exp(-n |\lambda_i - \lambda_j|) \qquad \text{for } 1 \leqq i, j \leqq d,$$

then b and b' are (n, δ)-close.

Actually we have

$$k^{-1}k' = k^{-1}xx^{-1}k' = \mathrm{Id} + k^{-1}x(x^{-1}(k' - k)).$$

If v_{ij} (respectively u_{ij}) is the general entry of $k^{-1}k'$ (respectively $k^{-1}x$), and if λ, μ are two distinct real numbers,

$$\begin{aligned}\sum_{\{(i,j) \mid \lambda_j = \lambda, \lambda_j = \mu\}} |v_{ij}|^2 &= \sum_{\{(i,j) \mid \lambda_i = \lambda, \lambda_j = \mu\}} \left| \sum_k u_{ik} w_{kj} \right|^2 \\ &\leqq \frac{1}{d^2} \sum_{\{(i,j) \mid \lambda_i = \lambda, \lambda_j = \mu\}} \sum_k |u_{ik}|^2 \exp(-2n |\mu - \lambda_k|) \\ &\leqq \frac{1}{d} \left(\sum_{\{k \mid \lambda_k \neq \lambda\}} e^{-2n|\lambda - \mu|} e^{2n\delta} + \sum_{\{k \mid \lambda_k = \lambda\}} e^{-2n|\lambda - \mu|} \right).\end{aligned}$$

This implies that k and k' are (n, δ) close and the claim follows by Lemma 2.

For any e in $E \cap U_{n,\delta}(x)$ the general entry u_{ij}^e of the matrix $x^{-1}k_e$ satisfies:

$$|u_{ij}^e| \leqq \exp(-n |\lambda_i - \lambda_j| + n\delta).$$

Therefore by the claim, for any i, j, $1 \leqq i, j \leqq d$, the sequence $\{u_{ij}^e, e \in E\}$ can take only $2de^{n\delta}$ different values. There are at most $(2de^{n\delta})^{d^2}$ different possible e in $E \cap U_{n,\delta}(x)$. This proves Lemma 3 with $C_1 = (2d)^{d^2}$, $\log C_2 = d^2$. ∎

LEMMA 4. *Let ν be a probability measure on B, $\delta > 0$, $\varepsilon > 0$. For n large enough, there is a set E_n with*

$$\nu(U_{n,\delta}(E_n)) \geqq 1 - \varepsilon,$$

$$\frac{1}{n} \log \operatorname{card} E_n \leqq \phi(B, \nu, \delta, \varepsilon) + \delta + \delta \log C_2 + \frac{1}{n} \log C_1.$$

PROOF. By definition of $\phi(B,\nu,\delta,\varepsilon)=\phi$, for n large enough, $\nu(E(\phi+\delta,n,\delta))\geqq 1-\varepsilon$. Let us choose E_n a maximal (n,δ) separated set in $E(\phi+\delta,n,\delta)$.

By maximality $U_{n,\delta}(E_n)$ contains $E(\phi+\delta,n,\delta)$ and thus $\nu(U_{n,\delta}(E_n))\geqq 1-\varepsilon$.

By definition of $E(\phi+\delta,n,\delta)$, we have:

$$\begin{aligned}\operatorname{card} E_n e^{-n(\phi+\delta)} &\leqq \sum_{x\in E_n}\nu(U_{n,\delta}(x))\\ &\leqq \int_B \operatorname{card}\{E_n\cap U_{n,\delta}(y)\}\nu(dy)\\ &\leqq C_1C_2^{n\delta}\qquad\text{by Lemma 3.}\end{aligned}$$

■

Lemma 4 will be used in the proof of Proposition 2 (see section 3.1); the following lemma is basic in the proof of Proposition 3. When $r>1$, we denote

$$\rho=\inf\{\lambda_{j_i}-\lambda_{j_{i+1}}\mid i=1,\dots,r-1\}.$$

(Remember Proposition 3 is trivial if $r=1$.)

LEMMA 5. *Let us suppose $r>1$ and consider $m>1$, $0<\chi<\rho/12$.*

Let u be some matrix, with column vectors $u_1,\dots,u_d$ such that for all $p=1,\dots,d$, all p-uples $0<i_1<i_2<\cdots<i_p\leqq d$,

$$\left|\frac{1}{m}\log\|u_{i_1}\wedge\cdots\wedge u_{i_p}\|-\sum_{i=1}^{p}\lambda_i\right|\leqq\chi.$$

Write $u=kan$ the Iwasawa decomposition of u, t_{ij} the general entry of an. There exists a constant C_3, such that for all (i,j) with $\lambda_i\geqq\lambda_j$

$$\left|\frac{1}{2m}\log\sum_{\{l\mid\lambda_l=\lambda_i\}}|t_{l,j}|^2-\lambda_i\right|\leqq 2\chi+\frac{C_3}{m}.$$

PROOF. We construct the Iwasawa decomposition and consider the matrix $t=k^{-1}u$.

It has the following properties:

$t_{ij}=0$ if $i>j$ and for all $p=1,\dots,d$, all $0<i_1<i_2<\cdots<i_p\leqq d$,

$$\|t_{i_1}\wedge\cdots\wedge t_{i_p}\|=\|u_{i_1}\wedge\cdots\wedge u_{i_p}\|.$$

This implies for $i\leqq j_1$, $\|t_i\|=\|u_1\|$ and therefore

$$\left|\frac{1}{m}\log\|t_i\|-\lambda_1\right|\leqq\chi.$$

Put $C(\chi) = (1 - e^{-(\rho - 3\chi)})^{-1}$ and consider $j > j_1$:

$$\frac{\| t_1 \wedge t_2 \wedge \cdots \wedge t_{j_1} \wedge t_j \|}{\| t_1 \wedge t_2 \wedge \cdots \wedge t_{j_1} \| \; \| t_j \|} \leqq e^{-m(\rho - 3\chi)}$$

and this means that most of the norm of t_j is achieved by the first j_1 coordinates. More precisely:

$$\left(\sum_{i=1}^{j_1} | t_{ij} |^2 \right) \Big/ \| t_j \|^2 \geqq 1 - e^{-2m(\rho - 3\chi)} \geqq C(\chi)^{-2}.$$

Hence:

$$\left| \frac{1}{2m} \log \sum_{i=1}^{j_1} | t_{ij} |^2 - \lambda_1 \right| \leqq \chi + \frac{\log C(\chi)}{m}$$

and this estimates the first j_1 lines of t.

Let $k = 1, \ldots, r-1$. The $(d - j_k \times d - j_k)$ matrix t' obtained by erasing the first j_k lines and columns of t has the following properties:

$$t'_{ij} = 0 \quad \text{if } i > j, \qquad j_k < i, j \leqq d$$

and for all $p = 1, \ldots, d - j_k$, all p-uple $(i_1, \ldots, i_p)$ with $j_k < i_1 < i_2 < \cdots < i_p \leqq d$,

$$\| t'_{i_1} \wedge \cdots \wedge t'_{i_p} \| = \frac{\| t_1 \wedge t_2 \wedge \cdots \wedge t_{j_k} \wedge t_{i_1} \wedge \cdots \wedge t_{i_p} \|}{\| t_1 \wedge t_2 \wedge \cdots \wedge t_{j_k} \|}.$$

In other words the matrix t' satisfies the same family of relations as t with exponents $\lambda_{i_k+1} \geqq \cdots \geqq \lambda_d$ and error 2χ.

We get in the same way for $j_k < j \leqq j_{k+1}$

$$\left| \frac{1}{2m} \log \sum_{i=j_k+1}^{j_{k+1}} | t_{ij} |^2 - \lambda_{j_k} \right| \leqq 2\chi$$

and for $j > j_k$

$$\left| \frac{1}{2m} \log \sum_{i=j_k+1}^{j_{k+1}} | t_{ij} |^2 - \lambda_{j_k} \right| \leqq 2\chi + \frac{\log C(2\chi)}{m}.$$

The lemma follows if $C_3 = -\log(1 - e^{-\rho/2})$. ∎

2.2. *Proof of Proposition* 3

We suppose in this section that μ is a probability measure on $\mathrm{GL}(d, \mathbf{R})$, satisfying $\int \log \| g \| \mu(dg) < +\infty$, $\int \log \| g^{-1} \| \mu(dg) < +\infty$, we consider the natural boundary (B,ν) and are going to prove that $\phi(B,\nu)\leqq\alpha(B,\nu)$. The idea of the proof is the same as in dimension 1 ([10] §III.4): a typical product $\omega_n\cdots\omega_0$ sends some set of fixed positive measure into some $U_{n,\delta}$ and $d\omega_n\cdots\omega_0\nu/d\nu$ is roughly $\exp(-n\alpha(B,\nu))$. This is possible only when $\phi(B,\nu)\leqq\alpha(B,\nu)$. To do this, we first apply Oseledeč' theorem, as stated in section 1.3 and we consider the map b from $E_1\subset\bar{\Omega}$ into B we defined there.

Lemma 6. *There exists a measurable map* $u: E_1\to \mathrm{GL}(d,\mathbf{R})$ *such that*
(i) $u(\bar{\omega})$ *belongs to the class of* $b(\bar{\omega})$,
(ii) *for all* p, $1\leqq p\leqq d$, *all* p*-uple* $0<i_1<i_2<\cdots<i_p\leqq d$,

$$\lim_{n\to+\infty}\frac{1}{n}\log\left\|\left(\overset{p}{\Lambda}(\bar{\omega}_n\cdots\bar{\omega}_0)\right)(u_{i_1}(\bar{\omega})\wedge\cdots\wedge u_{i_p}(\bar{\omega}))\right\|=\sum_{j=1}^{p}\lambda_j.$$

Proof. Remember that we choose in a measurable way a matrix $w(\bar{\omega})$ such that its column vectors are in the spaces $W^i(\bar{\omega})$. A matrix u satisfies (i) iff $u(\bar{\omega})=w(\bar{\omega})q(\bar{\omega})$ for some $q(\bar{\omega})$ in P. Choose some fixed matrix q in P such that for all p, $1\leqq p\leqq d$, the determinants of all the $(p\times p)$ square matrices made out of the first p lines of q are never zero. The matrix $u(\bar{\omega})=w(\bar{\omega})q$ satisfies also (ii). ∎

Fix $\bar{\omega}$ in E_1; the set of matrices satisfying Lemma 6(ii) is open in $\mathrm{GL}(d,\mathbf{R})$ and there is a smaller neighborhood of $u(\bar{\omega})$ where the convergences are uniform. The condition $\|v^{-1}\|<2\|u^{-1}(\omega)\|$ is also open and the projection from $\mathrm{GL}(d,\mathbf{R})$ into B is open. Consequently:

Corollary 1. *For all* $\bar{\omega}$ *in* E_1, *there exist a neighborhood* $O(\bar{\omega})$ *of* $b(\bar{\omega})$ *in* B *such that for* $\chi>0$, *there exists* $N(\bar{\omega},\chi)$ *and for all* b *in* $O(\bar{\omega})$ *there exists a matrix* v *in the class* b *satisfying* $\|v^{-1}\|<2\|u^{-1}(\omega)\|$ *and for all* $n\geqq N(\bar{\omega},\chi)$, *all* p, $1\leqq p\leqq d$, *all* p*-uple* $0<i_1<\cdots<i_p\leqq d$,

$$\left|\frac{1}{n}\log\left\|\left(\overset{p}{\Lambda}(\bar{\omega}_n\cdots\bar{\omega}_0)\right)(v_{i_1}\wedge\cdots\wedge v_{i_p})\right\|-\sum_{j=1}^{p}\lambda_j\right|\leqq\chi.$$

Lemma 7. *Fix* $\delta>0$, $\varepsilon>0$. *There exists an integer* N *such that for all* $n\geqq N$,

$$\bar{\mathbf{P}}(\{\bar{\omega}_{n-1}\cdots\bar{\omega}_0O(\bar{\omega})\subset U_{n,\delta}(b(\theta^{-n}\bar{\omega}))\})\geqq 1-\varepsilon/2.$$

Proof of Lemma 7 (compare with [10] lemma III.4.5.). We first choose N_1 by Corollary 1 such that $\bar{\mathbf{P}}(A_1)\geqq 1-\varepsilon/6$, where $A_1=\{\bar{\omega}\mid N(\bar{\omega},\delta/40d)\leqq N_1\}$. For $\bar{\omega}$ in A_1, b in $O(\bar{\omega})$ and $n\geqq N_1$ there exists a matrix v in the class b satisfying the conclusions of Corollary 1.

Choose also N_2 such that $\bar{\mathbf{P}}(A_2)\geqq 1-\varepsilon/6$, where

$$A_2 = \left\{ \bar{\omega} \mid \log 2 \| u^{-1}(\bar{\omega}) \| \leqq N_2 \frac{\delta}{10d} \right\}$$

and $u(\bar{\omega})$ is given by Lemma 6.

Write also a Cartan decomposition of the matrix $\bar{\omega}_n^{-1} \cdots \bar{\omega}_{-1}^{-1} = L_n(\bar{\omega})\Delta_n(\bar{\omega})K_n(\bar{\omega})$. By 1.3 there exists $n(\bar{\omega})$ such that if $n \geqq n(\bar{\omega})$ and $K(\bar{\omega})$ is any orthogonal matrix in the class $b(\bar{\omega})$,

$$\left| \frac{1}{n} \log \delta_j^{(n)}(\bar{\omega}) + \lambda_j \right| \leqq \frac{\delta}{10d},$$

$$\frac{1}{n} \log |u_{ij}^{(n)}(\bar{\omega})| \leqq -|\lambda_i - \lambda_j| + \frac{\delta}{10d},$$

where $\delta_j^{(n)}(\bar{\omega})$ are the diagonal entries of $\Delta_n(\bar{\omega})$, $u_{ij}^{(n)}(\bar{\omega})$ the entries of $K_n(\bar{\omega})K(\bar{\omega})$.

Choose N_3 such that $\bar{\mathbf{P}}(A_3) \geqq 1 - \varepsilon/6$ where $A_3 = \{\bar{\omega} \mid n(\bar{\omega}) \leqq N_3\}$. Choose now N bigger than N_1, N_2, N_3, $20C_3d/\delta$, and $(2\log C_4)/\delta$, $C_4 = d!\ d^{2d}$, and for $n \geqq N$ take $\bar{\omega}$ in $A_1 \cap A_2 \cap \theta^n A_3$.

Lemma 7 is proved if we show that for all b in $O(\bar{\omega}), \bar{\omega}_{n-1} \cdots \bar{\omega}_0$. b is (n, δ)-close to $b(\theta^{-n}\bar{\omega})$.

Choose v in b according to Corollary 1. We thus have to estimate the entries S_{ij} of the matrix $k^{-1}K(\theta^{-n}\bar{\omega})$, where k is the orthogonal term in the Iwasawa decomposition of $\bar{\omega}_{n-1} \cdots \bar{\omega}_0 v = kt$.

We get

$$\begin{aligned} k^{-1} &= tv^{-1}\bar{\omega}_0^{-1} \cdots \bar{\omega}_{n-1}^{-1} \\ &= tv^{-1}L_n(\theta^{-n}\bar{\omega})\Delta_n(\theta^{-n}\bar{\omega})K_n(\theta^{-n}\bar{\omega}). \end{aligned}$$

By Lemma 5, Corollary 1 and our choice of N_1 the entries t_{ij} of the matrix t satisfy $t_{ij} = 0$ for $i > j$ and $|t_{ij}| \leqq \exp n(\lambda_i + \delta/10d)$ for $i \leqq j$.

By Corollary 1 and our choice of N, the entries x_{ij} of $v^{-1}L_n(\theta^{-n}\bar{\omega})$ satisfy for all i, j

$$|x_{ij}| \leqq \exp n(\delta/10d).$$

By our choice of N, the estimations on $\delta_j^{(n)}$ and $u_{ij}^{(n)}$ are also valid. (Remember $\theta^{-n}\bar{\omega} \in A_3$.)

We get finally for $i > j$:

$$S_{ij} = \sum_{\substack{k \geqq i, \\ m = 1, \ldots, d}} t_{i,k} x_{k,m} \delta_m^{(n)}(\theta^{-n}\bar{\omega}) u_{m,j}^{(n)}(\theta^{-n}\bar{\omega})$$

and

$$|S_{ij}| \leqq \sum_{\substack{k \geqq i \\ m=1,\dots,d}} e^{n\lambda_i} e^{-n\lambda_m} e^{-n|\lambda_j - \lambda_m|} e^{4n\delta/10d}$$

$$\leqq d^2 e^{4n\delta/10d} e^{n(\lambda_i - \lambda_j)}$$

because $\lambda_m + |\lambda_j - \lambda_m| \geqq \lambda_j$ for all m. This gives estimates for the entries of $k^{-1}K(\theta^{-n}\bar{\omega})$ below the diagonal. Since it is an orthogonal matrix, we find for all entries $S_{i,j}$:

$$|S_{ij}| \leqq (d-1)!\,d^{2d} e^{4n\delta/10} e^{-n|\lambda_i - \lambda_j|}$$

$$\leqq \frac{1}{d} e^{n\delta} e^{-n|\lambda_i - \lambda_j|} \quad \text{by our choice of } C_4 \text{ and } N.$$

This proves that the matrices k and $K(\theta^{-n}\bar{\omega})$ are (n, δ)-close. By Lemma 2, this achieves the proof of Lemma 7. ∎

Lemma 8. *For $\bar{\mathbf{P}}$-a.e. $\bar{\omega}$,*

$$\alpha(B, \nu) \geqq \limsup_n -\frac{1}{n} \log \nu(\bar{\omega}_{n-1} \cdots \bar{\omega}_0 O(\bar{\omega})).$$

Proof of Lemma 8. We apply the ergodic theorem to the system $(\Omega \times B, \mathbf{P} \times \nu, \hat{\theta})$ and the function $-\log(d\omega_0^{-1}\nu/d\nu)(b)$. We get $\mathbf{P} \times \nu$ a.e. and in $L^1(\mathbf{P} \times \nu)$:

$$\alpha(B, \nu) = \lim_n -\frac{1}{n} \sum_{i=0}^{n-1} \log \frac{d\omega_i^{-1}\nu}{d\nu}(\omega_{i-1} \cdots \omega_0 \cdot b)$$

$$= \lim_n -\frac{1}{n} \log \frac{d\omega_0^{-1} \cdots \omega_{n-1}^{-1}\nu}{d\nu}(b).$$

Call E_2 the set of ω in Ω such that the sequence of classes of functions on B

$$\left\{ -\frac{1}{n} \log \frac{d\omega_0^{-1} \cdots \omega_{n-1}^{-1}\nu}{d\nu}(b),\ n > 0 \right\}$$

converges ν-a.e. and in $L^1(\nu)$ towards $\alpha(B, \nu)$. We have $\mathbf{P}(E_2) = 1$.

Now if $\pi(\bar{\omega}) \in E_2$ and $\nu(O(\bar{\omega})) > 0$, we have:

$$\alpha(B, \nu) = \lim_n -\frac{1}{n} \frac{1}{\nu(O(\bar{\omega}))} \int_{O(\bar{\omega})} \log \frac{d\omega_0^{-1} \cdots \omega_{n-1}^{-1}\nu}{d\nu}(b)\nu(db)$$

and therefore the conclusion of Lemma 8 is true for such an $\bar{\omega}$.

For $\bar{\mathbf{P}}$-a.e. $\bar{\omega}$, $\pi(\bar{\omega}) \in E_2$ because $\bar{\mathbf{P}} \circ \pi^{-1} = \mathbf{P}$, and for $\bar{\mathbf{P}}$-a.e. $\bar{\omega}$, $\nu((O(\bar{\omega})) > 0$

because $O(\bar{\omega})$ is a neighborhood of $b(\bar{\omega})$ and the law of b is ν. This achieves the proof of Lemma 8. ∎

Proposition 3 follows now clearly from Lemmas 8 and 7: fix $\delta > 0$, $\varepsilon > 0$. If n is large enough, we have on a set of probability bigger than $1-\varepsilon/2$

$$\nu(\bar{\omega}_{n-1}\cdots\bar{\omega}_0 O(\bar{\omega})) \geqq \exp(-n(\alpha(B,\nu)+\varepsilon))$$

and we also have on a set of probability bigger than $1-\varepsilon/2$, if n is large enough,

$$\bar{\omega}_{n-1}\cdots\bar{\omega}_0 O(\bar{\omega}) \subset U_{n,\delta}(b(\theta^{-n}\bar{\omega})).$$

In other words (remember the definition of ϕ) if n is large enough, there is a set of probability bigger than $1-\varepsilon$, such that $b(\theta^{-n}\bar{\omega})$ belongs to $E(\alpha(B,\nu)+\varepsilon, n, \delta)$. Since the law of $b(\theta^{-n}\bar{\omega})$ is also ν, this means exactly $\phi(B,\nu,\delta,\varepsilon) \leqq \alpha(B,\nu)+\varepsilon$. Proposition 3 follows from the arbitrariness of δ, ε.

III. Entropy of discrete groups

3.1. *Entropy estimate*

We suppose in this subsection that G is a discrete subgroup of $\mathrm{GL}(d,\mathbf{R})$ and we consider on $\mathrm{GL}(d,\mathbf{R})$ the following pseudo-distance d:

$$d(g_1,g_2) = \max(\log \|g_1 g_2^{-1}\|, \log \|g_2 g_1^{-1}\|).$$

We shall use discreteness of G only through the following property: there exists a constant C_5 such that a subset of $\mathrm{GL}(d,\mathbf{R})$ of diameter L contains less than C_5^L elements of G, if $L \geqq 1$.

Proposition 5. *Let G be a discrete subgroup of* $\mathrm{GL}(d,\mathbf{R})$, *$\mu$ a probability measure on G such that*

$$\sum_G \log \|g\| \mu(g) < +\infty, \qquad \sum_G \log \|g^{-1}\| \mu(g) < +\infty.$$

Then $-\Sigma \mu(g) \log \mu(g) < \infty$.

The proof of Proposition 5 is straightforward: call G_n the set of elements g of G such that $n \leqq d(e,g) < n+1$, and set $\mu_n = \mu(G_n)$. We have $\Sigma_n n\mu_n < +\infty$ and $H(\mu)$ can be bounded by

$$-\sum_n \mu_n \log \mu_n + \sum_n \mu_n \log(\operatorname{card} G_n) \leqq -\sum_n \mu_n \log \mu_n + \log C_5 \sum_n (n+1)\mu_n$$
$$< +\infty.$$
∎

By Proposition 5 we can define

$$h(G,\mu)=\inf_n \frac{1}{n} H(\mu^{(n)}).$$

LEMMA 9. *Let X be an orthogonal matrix, $\lambda_1 \geqq \cdots \geqq \lambda_d$ real numbers, $\delta > 0$, $n \geqq (\log d)/\delta$; there are less than $C_5^{10n\delta}$ elements of G which can be written $L\Delta K$, with L and K orthogonal, Δ diagonal, $K^{-1}(n,3\delta)$ close to X, and*

$$\left|\frac{1}{n}\log\delta_j+\lambda_j\right| \leqq \delta, \qquad j=1,\ldots,d.$$

PROOF. We have only to check that the diameter of the set of matrices $L\Delta K$ satisfying these properties is smaller than $10n\delta$. Let $g_1=L_1\Delta_1K_1$, $g_2=L_2\Delta_2K_2$ be two such matrices. We have

$$\begin{aligned}\|g_1g_2^{-1}\| &= \|L_1\Delta_1K_1K_2^{-1}\Delta_2^{-1}L_2^{-1}\|\\ &= \|\Delta_1K_1XX^{-1}K_2^{-1}\Delta_2^{-1}\|.\end{aligned}$$

The general entry of $K_1XX^{-1}K_2^{-1}$ is smaller than

$$\sum_{k=1}^{d}\exp[-n(|\lambda_i-\lambda_k|+|\lambda_k-\lambda_j|-6\delta)] \leqq \exp[-n(|\lambda_i-\lambda_j|-7\delta)]$$

and thus

$$\begin{aligned}\log\|g_1g_2^{-1}\| &\leqq 10n\delta+\sup_{i,j}[-n(|\lambda_i-\lambda_j|-\lambda_j+\lambda_i)]\\ &\leqq 10n\delta.\end{aligned}$$

The computation is the same for $\|g_2g_1^{-1}\|$. ■

We are going now to prove Proposition 2. We consider a probability measure μ on G such that $\Sigma_G \log\|g\|\mu(g) < +\infty$, $\Sigma_G \log\|g^{-1}\|\mu(g)<\infty$.

We apply 1.3 and consider the natural boundary (B,ν) and the product space $(\bar{\Omega},\bar{\mathbf{P}})$. We know that ν is the law on B of some variable b.

By Proposition 5 and the subadditive ergodic theorem, at $\bar{\mathbf{P}}$-a.e. $\bar{\omega}$:

$$\begin{aligned}\lim_{n\to+\infty} -\frac{1}{n}\log\mathbf{P}\{\omega' \mid \omega'^{-1}_{-n}\cdots\omega'^{-1}_{-1}=\bar{\omega}^{-1}_{-n}\cdots\bar{\omega}^{-1}_{-1}\} &= \lim_{n\to+\infty} -\frac{1}{n}\log\mu^{(n)}(\bar{\omega}_{-1}\cdots\bar{\omega}_{-n})\\ &= h(G,\mu).\end{aligned}$$

If we write again the Cartan decomposition of the matrix

$$\bar{\omega}^{-1}_{-n}\cdots\bar{\omega}^{-1}_{-1}=L_n(\bar{\omega})\Delta_n(\bar{\omega})K_n(\bar{\omega})$$

we have estimates on the diagonal terms of $\Delta_n(\bar{\omega})$ and of the (n,δ) closeness of $K_n^{-1}(\bar{\omega})$ and $X(\bar{\omega}) \in b(\bar{\omega})$ when n is large enough.

Fix $\delta > 0$, $0 < \varepsilon < \frac{1}{4}$. By Lemma 4 if n is large enough, there exists a set E_n with $\nu(U_{n,\delta}(E_n)) \geqq 1-\varepsilon$ and

$$\operatorname{card} E_n \leqq C_1 C_2^{n\delta} e^{n\delta} e^{n\phi(B,\nu)}.$$

Putting all these properties together, we have if n is large enough, $\bar{\mathbf{P}}(\Omega_i(n)) \geqq 1-\varepsilon$, $i = 1,2,3,4$, where:

$$\Omega_1(n) = \left\{ \begin{array}{c} \bar{\omega} \mid \text{the diagonal terms } \delta_j^{(n)}(\bar{\omega}) \text{ of } \Delta_n(\bar{\omega}) \text{ satisfy} \\ |(1/n)\log \delta_j^{(n)}(\bar{\omega}) + \lambda_j| \leqq \delta,\ j = 1,\dots,d \end{array} \right\},$$

$$\Omega_2(n) = \{\bar{\omega} \mid K_n^{-1}(\bar{\omega}) \text{ is } (n,\delta) \text{ close to any orthogonal matrix in } b(\bar{\omega})\},$$

$$\Omega_3(n) = \{\bar{\omega} \mid b(\bar{\omega}) \in U_{n,\delta}(E_n)\},$$

$$\Omega_4(n) = \{\bar{\omega} \mid \mu^{(n)}(\bar{\omega}_{-1} \cdots \bar{\omega}_{-n}) \leqq e^{-nh(G,\mu)} e^{n\delta}\}.$$

If $n \geqq (2\log d)/\delta$, for any $\bar{\omega}$ in $\Omega_2(n) \cap \Omega_3(n)$, there exists some x in E_n such that $K_n^{-1}(\bar{\omega})$ is $(n, 3\delta)$ close to any orthogonal matrix in x.

Fix x in E_n, X orthogonal matrix in x and call Ω_x the set of $\bar{\omega}$ in $\bigcap_i \Omega_i(n)$ such that $K_n^{-1}(\bar{\omega})$ is $(n,3\delta)$ close to X.

By Lemma 9, $\bar{\mathbf{P}}(\Omega_x) \leqq C_5^{10n\delta} e^{-nh(G,\mu)} e^{n\delta}$. Therefore we have, if n is large enough,

$$\begin{aligned} 1-4\varepsilon &\leqq \bar{\mathbf{P}}\Big(\bigcap_i \Omega_i(n)\Big) \\ &\leqq \sum_{x\in E_n} \bar{\mathbf{P}}(\Omega_x) \\ &\leqq C_1 C_2^{n\delta} e^{n\delta} e^{n\phi(B,\nu)} C_5^{10n\delta} e^{-nh(G,\mu)} e^{n\delta}. \end{aligned}$$

We get by letting n go to infinity

$$\phi(B,\nu) - h(G,\mu) \geqq -\delta(2 + \log C_2 C_5^{10})$$

and Proposition 2 follows from the arbitrariness of δ.

3.2. *Kaimanovich–Vershik results and Proposition* 4

Consider a countable group G and a probability μ on G, $\bigcup_n \operatorname{supp} \mu^{(n)} = G$. We recall Kaimanovich–Vershik results [8] leading to Proposition 4.

To every (G,μ) space (S,δ) they associate its Radon–Nikodym transform. It is a compact metric (G, μ) space (E, λ) isomorphic as a measure space to the quotient of (S, ρ) by the measurable partition generated by the Radon–Nikodym derivatives Δg on S, $g \in G$:

$$\Delta g(s) = \frac{dg\rho}{d\rho}(s) \qquad \rho\text{-a.e.}$$

They also construct abstractly the Poisson boundary (Γ, ν), prove that

$$\alpha(S, \rho) \leqq h(G, \mu) \qquad \text{for all } (G, \mu) \text{ space } (S, \rho)$$

and that if $\alpha(S, \rho) = h(G, \mu)$, then the quotient map from (S, ρ) to its Radon–Nikodym transform (E, λ) factorizes through (Γ, ν) ([8] Theorem 3.2).

If (S, ρ) is a boundary, the function Δg, $g \in G$ separate ρ-a.e. point and thus the quotient map defining (E, λ) is an isomorphism.

Under the condition of Proposition 4, (S, ρ) is therefore isomorphic to (Γ, ν), in other words (S, ρ) is a Poisson boundary.

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